WORMHOLES IN $f(R)$ GRAVITY

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Introduction

Wormholes are topological solutions that join two points from the same or different universes, and are described by the shape function and the redshift function, which satisfy specific constraints. The shape function satisfies the following conditions:

- $b(r_0) = r_0$,
- $\left(1 - \frac{b(r)}{r}\right) > 0$, for $r > r_0$,
- $b'(r_0) < 1$ (flaring out condition), and
- $\lim_{r\to\infty} \frac{b(r)}{r} = 0$ (asymptotic flatness condition)

Further, the redshift function is always finite in order to prevent an indefinite gravitational tidal effect, therefore ensuring that the solution does not contain an event horizon. The Morris-Thorne wormhole metric is [**d**]:

$$ds^2 = -dt^2\chi(r) + dr^2 B(r) + r^2 d\Omega^2 \tag{1}$$

Where $B(r)$ contains the shape function and $\chi(r)$ is the redshift function, which may be normalised to be unity. Wormhole solutions would naturally violate the energy conditions, since the energy condition parameters would be negative throughout the solution due to exotic matter, which is required to hold the wormhole open. The energy conditions may be stated as:

- Null energy condition: $\rho + p_{r,t} \geq 0$,
- Strong energy condition: $\rho + p_r + 2p_t \geq 0$,
- Weak energy condition: $\rho \geq 0$, $(NEC)_{r,t} \geq 0$,
- Dominant energy condition: $\rho \geq 0$, $\rho - |p_{r,t}| \geq 0$

Where $p_{r,t}$ refers to the radial/tangential components of the energy condition parameters. The energy-momentum tensor for fluids would be given by

$$T_{\mu\nu} = (\rho + p_t)u_{\mu\nu} - p_t g_{\mu\nu} + (p_r + p_t)X_\mu X_\nu \tag{2}$$

Recent research of wormhole solutions has shown that using corrections to the Einstein-Hilbert action to derive modified theories of gravity such as $f(R)$, $f(R,T)$ and Lovelock gravity, wormhole solutions can be shown to be physically viable, unlike in general relativity, where the energy conditions are totally violated. In the $f(R)$ background, we consider different forms of $f(R)$ models along with $b(r)$ and $\chi(r)$ for determining wormhole solutions.The basic research of wormholes in GR has a feature of violating the energy conditions. This is because a negative energy-density is an elementary complication of wormholes. In $f(R)$ gravity, we can instead ignore this feature and define the physical properties of wormholes using a purely geometric correction. One of the first re-search showed that this was possible was by Starobinskii [extract from[**a**]].

$f(R)$ gravity

$f(R)$ gravity is obtained by extending the Einstein-Hiflbert action of general relativity into an arbitrary class of functions defined based on the Ricci scalar:

$$S_{f(R)} = \frac{1}{2k}\int f(R)\, d^4x\sqrt{-g} \tag{3}$$

Note that no matter terms are perturbed in this class of theories of gravity – if we introduced functions in the matter action containing the trace of the energy-momentum tensor, we would obtain $f(R,T)$ theories of gravity, defined by an action

$$S = \frac{1}{2k}\int f(R)\, d^4x\sqrt{-g} + \int f(T)\, d^4x\sqrt{-g} \tag{4}$$

We will only consider the first set of theories, i.e. $f(R)$ theories of gravity. The Einstein field equations for the metric would be in terms of ρ, p_t and p_r as follows:

$$\rho + F\frac{b'(r)}{r^2} - \left(1 - \frac{b(r)}{r}\right)F'\chi'(r) - \mathcal{H} \tag{5}$$

$$p_t = F\frac{b(r) - rb'(r)}{2r^3} - \frac{F'}{r}\left(1 - \frac{b(r)}{r}\right) + F\left(1 - \frac{b(r)}{r}\right) \left(\chi''(r) - \frac{(rb'(r)\chi'(r))}{2r^2 - 2rb(r)} + \frac{\chi'(r)}{r} + \chi'^2(r)\right) + \mathcal{H} \tag{6}$$

$$p_r = -F\frac{b(r)}{r^3} + 2\left(1 - \frac{b(r)}{r}\right)\frac{\chi'(r)F}{r} - \left(1 - \frac{b(r)}{r}\right) \left(F'' - \frac{F'(rb'(r) - b(r))}{2r^2 - 2r^2\left(1 - \frac{b(r)}{r}\right)}\right) + \mathcal{H} \tag{7}$$

Where $\mathcal{H} = \frac{1}{4}(FR + \Box F + T)$, $F(R)$ is $\mathrm{d}f(R)/\mathrm{d}R$ and the prime denotes $\partial/\partial r$. [extract from[**a**]]. Consider the case of Nojiri-Odintsov $f(R)$ theories, defined by [**h**]

$$f(R) = R + \alpha R^m - \beta R^{-n}$$

Also, we will consider the wormhole solution to be described by a shape function [**g**]

$$b(r) = r_0\left(\frac{x^r}{x^{r_0}}\right)$$

Selecting all the parameters $\alpha = \beta = 0$, we would obtain the usual general relativity equations (5-7), and the energy condition parameters would clearly be all negative, implying that the solution contains exotic matter. Further, the anisotropy parameter would also indicate a repulsive geometry.
However, by selecting certain values of the base x and the parameters α and β, it is possible to find a wormhole solution that has a variational set of energy condition parameters, i.e. the energy condition parameters are neither totally violated nor totally preserved. Due to this, the regions where the exotic matter maybe found can be understood, which allows us to further understand the nature of the wormhole solution. The following is an extract from [**a**] that shows a variational energy conditions parameter solution:

	NEC	SEC	DEC	Anisotropy parameter
Radial	Preserved for r>l_r^1	Violated for all $r > l_r$	Preserved for $r > l_r^1$	
Tangential	Preserved for $r > l_r^2$	Violated for all $r > l_r$	Preserved for all $r > l_r^2$	
				Negative for all $r > l_r^1$

General format of $f(R)$ from the shape function

Consider we have a barotropic model:

$$p_t = \omega\rho \tag{8}$$

The way we can find the general format of $f(R)$ from a chosen shape function is by using the following steps:

- We consider the shape function we want for the wormhole solution,
- We then solve the field equations (5-7) to find the individual energy conditions parameters,
- This also gives us the $f(R)$ for the shape function, referring to the primed value of $f(R)$

We will consider an example to illustrate this. Considering the Samanta et al [**i**] form of shape function $b(r) = re^{-2(r-r_0)}$. For the considered equation of state, we can find the value of $F(R)$ as

$$F = (1 - \exp(-2(r - r_0)))^{\frac{2\omega+1}{2}} \times \exp\left(\int L(r)\, dr\right) \tag{9}$$

The energy conditions components can be found out to be:

$$\rho = A(1 - A)^{\omega+\frac{1}{2}}\frac{1 - 2r}{r^2} \times \exp\int (L(r))\, dr \tag{10}$$

$$p_t = \omega\left(\frac{1 - 2r}{r^2}\right)(1 - A)^{\omega+\frac{1}{2}} \times \exp(\int L(r)\, dr) \tag{11}$$

$$p_r = -\rho\left[\frac{1}{1 - 2r}\left(1 + \frac{r^2(r(1 + 2\omega) - \omega)}{r \times B - 1}\right) + \frac{r^2 \times B}{1 - 2r} \left(\frac{(1 + 2\omega)^2 - 2(1 + 2\omega)B}{(B - 1)^2} + \frac{\omega^2 + \omega((2r + 1) \times B - 1)}{r^2(B - 1)^2} + \frac{2\omega(1 + 2\omega)}{r(B - 1)^2}\right.\right. \tag{12}$$

Note that the Ricci scalar is $R = \frac{2b'(r)}{r^2}$, which for our shape function (13) gives $R = \frac{2(1-2r)\exp(2(r-r_0))}{r^2}$. We then see that $f(R)$ is of the form

$$f(R) = \frac{F}{2}\left[\frac{2(1 - 2r) \times B}{r^2} + 2(1 - A)\left(\frac{(1 + 2\omega)^2 - 2(1 + 2\omega \times B)}{(B - 1)^2} + \frac{\omega^2 + \omega((2r + 1) \times B - 1)}{r^2(B - 1)^2} - \frac{2(1 +}{r(B -} \right.\right. \frac{r(1 + 2\omega) - \omega}{r(B - 1)}\left(\frac{(-r + 2(B - 1))}{r \times B}\right) - \frac{1 - 2r}{r^2}(1 - 2\omega) \times A + \frac{1}{r^2 \times B} + A\frac{(1 +}{r(} \tag{13}$$

This $f(R)$ again has a variational energy conditions set, i.e. the energy conditions are violated in selected components.

We can also find the embedding of the wormholes, which can be done by considering a hypersurface of the topology by setting $t =$ const. and $\theta = \pi/2$. The resulting metric would be

$$ds^2_{\mathcal{H}} = dr^2\left(1 - \frac{b(r)}{r}\right)^{-1} + r^2 d\phi^2(1)^2 \tag{14}$$

This would further result in the cylindrical geometry, and using this we can find the embedding diagram for the wormhole solution.

$$ds^2_{\mathcal{H}} = \left(1 + \left(\frac{dz}{dr}\right)^2\right)dr^2 + r^2 d\phi^2(1)^2 \tag{15}$$

Conclusions

Wormholes that satisfy the energy conditions can be found out using modified theories of gravity, which use geometric corrections to allow a physically viable description of wormhole solutions. Since wormholes in the general relativity background are not physically viable, it is necessary to understand how such modified theories of gravity can be experimentally seen to be true so that the resulting forms of modified gravity can be applied to such solutions. $f(R)$ theories have so far been investigated in terms of wormhole solutions without charge, and such theories usually violate one or more energy condition components or are variationally true. It is therefore important to describe a totally viable wormhole solution, i.e. wormhole solutions that preserve all the energy conditions throughout the solution. Further investigations of $f(R,T)$ gravity to describe wormholes will be published soon, with focus on charged wormhole solutions and the general format of $f(R,T)$ for a chosen wormhole shape function.

References

References

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